Preservation of the Discrete Geostrophic Equilibrium in Shallow Water Flows

E. Audusse, R. Klein, D.D. Nguyen, and S. Vater

Abstract We are interested in the numerical simulation of large scale phenomena in geophysical flows. In these cases, Coriolis forces play an important role and the circulations are often perturbations of the so-called geostrophic equilibrium. Hence, it is essential to design a numerical strategy that preserves a discrete version of this equilibrium. In this article we work on the shallow water equations in a finite volume framework and we propose a first step in this direction by introducing an auxiliary pressure that is in geostrophic equilibrium with the velocity field and that is computed thanks to the solution of an elliptic problem. Then the complete solution is obtained by working on the deviating part of the pressure. Some numerical examples illustrate the improvement through comparisons with classical discretizations.

Keywords Geostrophic Adjustment, Shallow Water Flows, Finite Volume Method, Well-balanced Scheme
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1 Introduction

We are interested in the numerical simulation of large scale phenomena in geophysical flows. At these scales, Coriolis forces play an important role and the atmospheric or oceanic circulations are frequently observed near geostrophic equilibrium situations, see for example [11, 12]. For this reason it is essential to design a numerical strategy that preserves a discrete version of this geostrophic

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equilibrium: if numerical spurious waves are created, they quickly become higher than the physical ones we want to capture. This phenomenon is well known but its solution in the context of finite volume methods is still an open problem. We address this question in this article.

One of the most popular systems that is used to model such quasi-geostrophic flows are the shallow water equations with β-plane approximation

$$h_t + \nabla \cdot (h\overline{u}) = 0, \tag{1}$$

$$(h\overline{u})_t + \nabla \cdot (h\overline{u} \otimes \overline{u}) + \nabla(\frac{gh^2}{2}) = -f\,\mathbf{e}_z \times (h\overline{u}), \tag{2}$$

The shallow water system is the simplest form of equations of motion that can be used to model Rossby and Kelvin waves in the atmosphere or ocean, and the use of the β-plane approximation allows the model to take into account a non-constant Coriolis parameter f that varies linearly with the latitude without considering a spherical domain. We choose to work in a finite volume framework to discretize the equations because of its ability to deal with complex geometries and its inherent conservation property, see [4, 9]. In this context, the discrete preservation of the geostrophic equilibrium, which is mainly the balance between pressure gradient and Coriolis forces in (2), is a hard touch: the main reason is that the fluxes are upwinded for stability reasons while the source terms are usually discretized in a centered way.

The question of the preservation of non-trivial equilibria in geophysical fluid models has received great attention in the area of numerical modeling in the last decade. Many studies were devoted to the preservation of the so-called hydrostatic and also lake-at-rest equilibria, see [2–4] and references therein. More recently some authors investigated the problem of the geostrophic equilibrium [5, 6, 8, 10]. However, this question is more delicate for two reasons: it is an essentially 2d problem, and it involves a non-zero velocity field. It follows that its solution is still incomplete. In this work we propose a solution to this problem by introducing an auxiliary water depth which is in geostrophic balance with the velocity field and then by working on the deviation between the actual and auxiliary water depths instead of considering the water depth itself. The auxiliary water depth is computed through the solution of a Poisson problem on a dual grid [13].

2 Position of the problem

In this short note we present the method by considering a constant Coriolis parameter. In order to exhibit the importance of the geostrophic equilibrium, we introduce the non-dimensional version of the shallow water equations (1)–(2) written in non-conservative form

$$h_t + \nabla \cdot (h\overline{u}) = 0,$$

$$\overline{u}_t + \overline{u} \cdot \nabla \overline{u} + \frac{1}{\mathbf{Fr}^2}\nabla h + \frac{1}{\mathbf{Ro}} 2\mathbf{e}_z \times \overline{u} = 0.$$

Here, h and $\overline{u}$ are the unknown dimensionless depth and velocity fields and

$$\mathbf{Fr} = \frac{\overline{U}}{\sqrt{gH}}, \qquad \mathbf{Ro} = \frac{\overline{U}}{\Omega L}$$

are the Froude and Rossby numbers, respectively, with $\overline{U}$, L and H some characteristic velocity, length and depth for the flow, g the gravity coefficient and Ω the angular velocity of the earth. For large scale phenomena typical values for these numbers are

$$\mathbf{Fr} \approx \mathbf{Ro} \approx \epsilon = 10^{-2},$$

We then expand the unknowns in term of ε

$$h = h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots, \qquad \overline{u} = \overline{u}_0 + \varepsilon \overline{u}_1 + \varepsilon^2 \overline{u}_2 + \ldots$$

and we keep the leading order terms to exhibit the following stationary state

$$O\left(\epsilon^{-2}\right) : \nabla h_0 = 0 \tag{3}$$

$$O\left(\epsilon^{-1}\right) : \nabla h_1 + 2e_z \times \overline{u}_0 = 0 \tag{4}$$

$$O\left(\epsilon^{0}\right) : \nabla \cdot \overline{u}_0 = 0, \tag{5}$$

This set of equations is called the geostrophic equilibrium. It follows from equation (3) that the water depth is constant at the leading order and from equation (5) that the main part of the velocity field is divergence free. Equation (4) is nothing but the fact that the pressure gradient and the Coriolis term are in balance for leading varying terms h_1 and $\overline{u}_0$. Let's now turn to the numerical point of view. Preservation of the discrete equilibrium (3) is obvious. The divergence free condition (5) is much more delicate to deal with but it has been widely investigated for Stokes or Navier-Stokes equations, mostly in the framework of finite element methods. It is also the subject of a recent work [13], where the authors study the zero Froude number limit of the shallow water equations. In this note we focus on a proper way to preserve the balance in equation (4).

3 The well-balanced finite volume scheme

We choose to discretize the shallow water equations (1)–(2) in a finite volume framework [4, 9]. The reason to consider this particular method is related to its inherent conservation properties that are interesting for geophysical applications and in particular for long time simulations [1]. A second reason is that the finite volume method is also able to deal with sharp fronts that can occur in geophysical

applications. We first recall the formulation of the finite volume method and the classical centered discretization of the Coriolis source term. Then, we derive the new well-balanced scheme by introducing an auxiliary pressure that is computed through the solution of a Laplace equation on a dual grid.

System (1)–(2) is a particular case of a 2d conservation law with source term:

$$U_t + (F(U))_x + (G(U))_y = S(U), \tag{6}$$

in which $U = (h, hu, hv)^T$ and

$$F(U) = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{pmatrix}, \; G(U) = \begin{pmatrix} hv \\ huv \\ hv^2 + \frac{1}{2}gh^2 \end{pmatrix}, \; S(U) = \begin{pmatrix} 0 \\ 2\Omega hv \\ -2\Omega hu \end{pmatrix}.$$

In this note we only consider Cartesian grids. Then, the finite volume discretization of equation (6) leads to the computation of approximated solutions $U^n_{i,j}$ through the discrete formula

$$U^{n+1}_{i,j} = U^n_{i,j} - \frac{\delta t}{\delta x}\left(\mathbf{F}^n_{i+\frac{1}{2},j} - \mathbf{F}^n_{i-\frac{1}{2},j}\right) - \frac{\delta t}{\delta y}\left(\mathbf{G}^n_{i,j+\frac{1}{2}} - \mathbf{G}^n_{i,j-\frac{1}{2}}\right) + \delta t\, S^n_{i.j},$$

where $\mathbf{F}^n_{i+\frac{1}{2},j}$ is a discrete approximation of the flux $F(U)$ along the interface between cells $C_{i,j}$ and $C_{i+1,j}$ that is constructed through a three points formula

$$\mathbf{F}^n_{i+\frac{1}{2},j} = \mathscr{F}\left((h^n_{i,j}, u^n_{i,j}, v^n_{i,j}), (h^n_{i+1,j}, u^n_{i+1,j}, v^n_{i+1,j})\right). \tag{7}$$

Here we use the HLL solver [7] to compute these approximations.

The classical discretization of the source term $S^n_{i.j}$ is computed through the centered formula

$$S^n_{i.j} = \begin{pmatrix} 0 \\ -2\Omega_z \times (h^n_{i,j}\overline{u}^n_{i,j}) \end{pmatrix}$$

where $h^n_{i,j}$ denotes the approximated value at time t^n on cell $C_{i,j}$. We will exhibit in the last section that this approach suffers from important drawbacks when we consider applications for small Froude and Rossby numbers.

The main idea of our method to overcome this problem is to introduce an auxiliary water depth h_c that is in balance with Coriolis forces related to the actual velocity field. This idea is an extension of the notion of hydrostatic reconstruction that was introduced in [3] for the Euler equations and in [2] for shallow water flows. Here, h_c will satisfy the equation

$$g\nabla h_c = -2\Omega \times \overline{u}. \tag{8}$$

In our approach, h_c is discretized as a grid function, which is piecewise bilinear on each grid cell and continuous at the interfaces. The second ingredient of the well-balanced scheme is the representation of the Coriolis forces by the gradient of this quantity. Furthermore, the fluxes in the conservative part of the scheme are modified in the following way: For each cell, we introduce a deviation in the water depth by

$$\Delta h^n_{i,j} = h^n_{i,j} - h^n_c(x_i, y_j)$$

Then, the interface water depths are computed by

$$\widehat{h}^{n,k,x}_{i+\frac{1}{2},j} = \frac{1}{2}\left[h^n_c\left(x_{i+\frac{1}{2},j+\frac{1}{2}}\right) + h^n_c\left(x_{i+\frac{1}{2},j-\frac{1}{2}}\right)\right] + \Delta h^n_{k,j}, \quad \text{for} \quad k = i, i+1.$$

and the original three points formula (7) for the flux is replaced by

$$\mathbf{F}^n_{i+\frac{1}{2},j} = \mathscr{F}\left((\widehat{h}^{n,i,x}_{i+\frac{1}{2},j}, u^n_{i,j}, v^n_{i,j}), (\widehat{h}^{n,i+1,x}_{i+\frac{1}{2},j}, u^n_{i+1,j}, v^n_{i+1,j})\right),$$

If the flow satisfies the geostrophic equilibrium, $\hat{h}$ and h_c are equal. The consistency of the flux will then provide some numerical balance between the conservative part and the source term that will directly impact the results. Note also that the time step is now related to the interface water depths. Nevertheless, in the numerical applications, the numerical values remain very close for both methods.

It remains to explain the computation of the auxiliary water depth h_c that is the solution of a discrete equivalent of equation (8). We first take the divergence of this equation and then search for the solution of a Poisson equation

$$-\Delta\phi = \nabla \cdot \left(\overline{k} \times \overline{u}\right).$$

Integration of this equation on the dual cell $C_{i+\frac{1}{2},j+\frac{1}{2}}$ and application of the Gauss theorem leads to

$$\int_{\partial C_{i+\frac{1}{2},j+\frac{1}{2}}} \nabla\phi \cdot \overline{n}\, \mathrm{d}\sigma = -\int_{\partial C_{i+\frac{1}{2},j+\frac{1}{2}}} \overline{k}_z \overline{u} \cdot \overline{t}\, \mathrm{d}\sigma,$$

where $\overline{n}$ (resp. $\overline{t}$) is a normal (resp. tangential) vector to the interface of the dual cell. We solve this equation by using the technique presented in [13] for the solution of a similar problem. We refer the reader to this article for the details of the method that is in particular proved to provide an inf-sup-stable projection. We finally obtain a linear system with a nine point stencil. The boundary conditions for this auxiliary problem are prescribed by using the fact that the computed pressure (or height) field is equivalent to a stream function for the associated balanced geostrophic flow. For example a rigid wall type boundary condition for the flow translates into a Dirichlet type boundary condition for the stream function. Similar types of equivalences can be used to prescribe other types of boundary conidtions.

4 Numerical results

In order to test the new scheme, we consider a stationary vortex in the square domain $[0, 1] \times [0, 1]$. We consider periodic boundary conditions, and as initial conditions we choose a velocity field of the form

$$\overline{u}_0(r, \theta) = v_\theta(r)\overline{e}_\theta, \quad v_\theta(r) = \varepsilon \left[5r \ \chi\left(r < \frac{1}{5}\right) + (2 - 5r)\, \chi\left(\frac{1}{5} \leq r < \frac{2}{5}\right)\right],$$

where r is the distance to the center of the domain and χ denotes the characteristic function of a given interval. Some computations show that the vortex is a stationary solution of the shallow water equations (1)-(2), if the initial water depth $h_0(r)$ is a radial solution of the ODE

$$h_0'(r) = \frac{1}{g}\left(2\Omega v_\theta + \frac{v_\theta^2}{r}\right).$$

Note that if we choose a water depth and an angular velocity of order $O(1)$, the Froude and Rossby numbers are of order $O(\epsilon)$. It follows that our interest is for small values of the parameter ϵ.

We first work on a regular grid with 30×30 cells and we consider four Froude resp. Rossby numbers: $0.05, 0.1, 0.5$ and 1. The numerical solution is computed by using both schemes described in the previous section. In order to compare the accuracy of the schemes, we compute the relative L^2 error in the water depth. In Fig. 1 we present the time evolution of this error for the four values of ϵ. It appears that for both schemes the error is increasing with time before reaching a stationary value. More interesting is that for the classical (resp. well-balanced) scheme the error is increasing (resp. decreasing), when the Froude number is decreasing. While the error is of the same order for both schemes when $\epsilon = 1$, for other values of ϵ the well-balanced scheme is always more precise than the classical one.

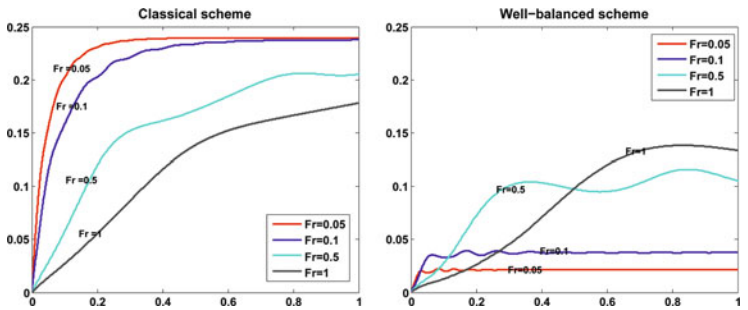

Fig. 1 Error in time for both classical and well-balanced schemes (30×30 grid cells) and for four different Froude numbers

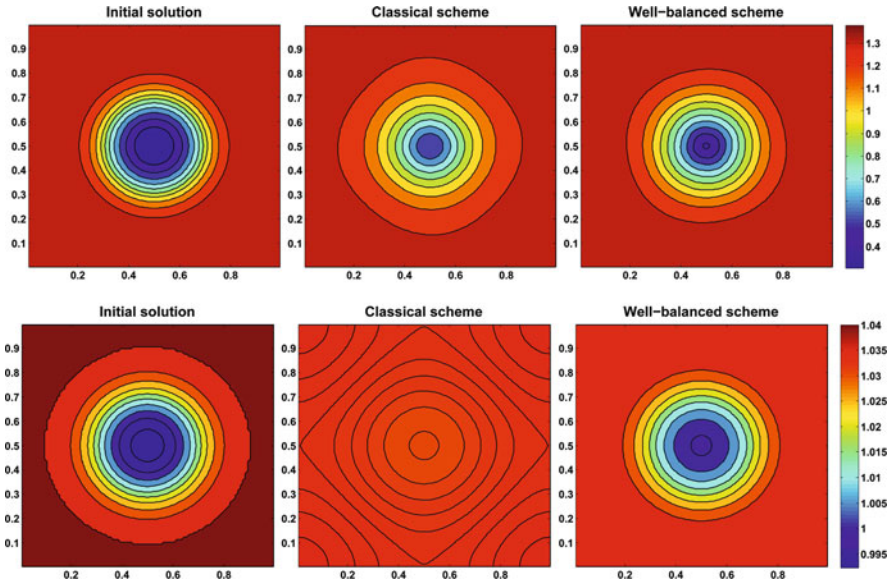

Fig. 2 Contour of the computed fluid depth with 100×100 grid cells. $\mathbf{Fr} = 0.95$ (Top), $\mathbf{Fr} = 0.1$ (bottom)

We then consider a finer grid with 100×100 cells and we present the water depth for both schemes and for two values of ϵ: 0.95 (large) and 0.1 (small). In Fig. 2 we present the 2d contour of the water depth. The results look similar and quite close to the initial solution when ϵ is large (top row). But when ϵ is small (bottom row), the classical scheme totally fails to compute the right solution, whereas the water depth computed by the well-balanced scheme stays close to the initial one. In Fig. 3 we give more quantitative results by presenting a cut of the solution along x–axis at $y = 0.5$. These pictures clearly exhibit that the results are very close when ϵ is large, but very different when ϵ is small. In this last case the classical scheme is not able to maintain the vortex, whereas the well-balanced scheme preserves the shape of the free surface. Note that the small diffusion that is observed even for the well-balanced scheme is due to the fact that we consider only first order schemes in this work. We end this short note by some words on the CPU time. We first notice that for the last numerical test case, the time steps are very close for both methods, as it is reported in the table below. We then consider the CPU time for both methods and conclude that it is four times larger for the well-balanced scheme. It is obviously due to the solution of the linear system related to the elliptic problem at each time step. This observation leads to two comments. First, and since the solution of the linear system is only required for the computation of the auxiliary water depth, it is possible to obtain a compromise between accuracy and efficiency of the whole process by considering iterative methods with a small number of iterations. Second we recall that our final objective is to couple the presented process with a numerical

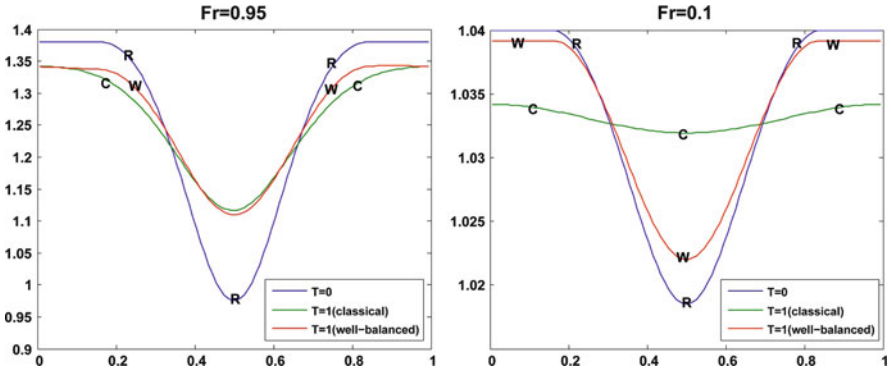

Fig. 3 Fluid depth profiles – cut along the x–axis at $y = 0.5$ with 100×100 grid cells. R line: Initial solution, W line: Well-balanced scheme, C line: Classical scheme

scheme adapted for small Froude number flows and then to generalize the method presented in [13] to rotating flows. Since the technique introduced in [13] already requires for the solution of a related linear system, the additional computational cost of the well-balanced process presented here is very small.

	Classical scheme	Well-balanced scheme
Time Step	9.7702e-005	9.7464e-005
CPU Time	1547 s	5564 s

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The paper is in final form and no similar paper has been or is being submitted elsewhere.